10.1. Cut points and injectivity radius. Let M be a complete Riemannian manifold, and let $p \in M$.

(a) Show that if $\bar{p} \in M$ is a cut point of p along the unit speed geodesic $c\colon [0,l] \to M$, then p is a cut point of $\bar{p}$ along $\bar{c}\colon [0,l] \to M$, $\bar{c}(t) := c(l-t)$.

(b) Show that $\mathrm{inj}_p := d(p, \mathrm{Cut}(p))$ equals the supremum of all $r > 0$ such that $\exp_p|_{B_r}$ is injective, and also the supremum of all $r > 0$ such that $\exp_p|_{B_r}\colon B_r \to B(p,r)$ is a diffeomorphism.

10.2. Non-negative Ricci curvature and maximal volume. Show that a complete Riemannian m-manifold M with $\mathrm{Ric} \geq 0$ and

$$\lim_{r\to\infty} \frac{\mathrm{vol}(B(p,r))}{V_{m,0}(r)} = 1$$

for some $p \in M$ is isometric to $\mathbb{R}^m$.

10.3. Growth of finitely generated groups.

(a) Verify that for a free group on $k \geq 2$ generators, with generating set $A = \{a_1, \ldots, a_k\}$, the growth function satisfies

$$w_A(r) = \frac{k(2k-1)^r - 1}{k-1}$$

and hence $(2k-1)^r \leq w_A(r) \leq (2k+1)^r$ for all integers $r \geq 1$.

(b) Let $\tilde{M} = G$ be the nilpotent Lie group consisting of all 3×3 upper triangular real matrices with 1's on the diagonal (the *Heisenberg group*), and let Γ be the subgroup of all integer matrices. Then the quotient space $\tilde{M}/\Gamma$ is a compact 3-dimensional manifold with fundamental group Γ. Show that Γ is generated by $A = \{x, y\}$, where $x = I + e_{12}$ and $y = I + e_{23}$, and there exist constants $c_2 \geq c_1 > 0$ such that

$$c_1 r^4 \leq w_A(r) \leq c_2 r^4$$

for all integers $r \geq 0$. (Theorem 6.11 and Theorem 6.12 then show that $\tilde{M}/\Gamma$ does neither admit a metric with $\sec < 0$ nor a metric with $\mathrm{Ric} \geq 0$.) See the outline of Lemma 4 in J. Milnor, A note on curvature and fundamental group, J. Differential Geometry 2 (1968), 1–7.